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Spin orbit coupling for explanation of Magic Numbers: Nuclear Spin and Parity

From the shell model it is clear that there certain numbers **2,8,20,28,50,82 and 126** and if the number of nucleons is equal to those number. The nucleus shows extraordinary stability. Since neither the infinite well potential nor the harmonic oscillator potential is able to explain the magic numbers. So in order to predict the higher magic numbers, we need to take into account other interactions between the nucleons. The first interaction we analyze is the spin-orbit coupling. The associated potential can be written as

$$\frac{1}{\hbar^2} V_{so}(r) \hat{\vec{l}} \cdot \hat{\vec{s}}$$

$\vec{s}$ and $\vec{l}$ are spin and angular momentum operators for a single nucleon. This potential is to be added to the single-nucleon mean-field potential. As nucleon interaction forces are spin dependent. This type of interaction motivates the form of the potential above (which again is to be taken in a mean-field picture). We can calculate the dot product with the same trick already used:

$$\left\langle \hat{\vec{l}} \cdot \hat{\vec{s}} \right\rangle = \frac{1}{2}(\hat{\vec{j}}^2 - \hat{\vec{l}}^2 - \hat{\vec{s}}^2) = \frac{\hbar^2}{2}[j(j+1) - l(l+1) - \frac{3}{4}]$$

where $\hat{\vec{j}}$ is the total angular momentum for the nucleon. Since the spin of the nucleon is $s = \frac{1}{2}$, the possible values of j are $j = l \pm \frac{1}{2}$. Then $j(j+1) - l(l+1) = (l \pm \frac{1}{2})(l \pm \frac{1}{2} + 1) - l(l+1)$, and we obtain

$$\left\langle \hat{\vec{l}} \cdot \hat{\vec{s}} \right\rangle = \begin{cases} l\frac{\hbar^2}{2} & \text{for j=l+}\frac{1}{2} \\ -(l+1)\frac{\hbar^2}{2} & \text{for j=l-}\frac{1}{2} \end{cases}$$

and the total potential is

$$V_{nuc}(r) = \begin{cases} V_0 + V_{so}\frac{l}{2} & \text{for j=l+}\frac{1}{2} \\ V_0 - V_{so}\frac{l+1}{2} & \text{for j=l-}\frac{1}{2} \end{cases}$$

Now recall that both V_0 is negative and choose also V_{so} negative. Then:

- When the spin is aligned with the angular momentum ($j = l + 1/2$) the potential becomes more negative, i.e. the well is deeper and the state more tightly bound.
- When spin and angular momentum are anti-aligned the system's energy is higher.

The energy levels are thus split by the spin-orbit coupling (see figure 1). This splitting is directly proportional to the angular momentum ***l*** (is larger for higher ***l***): $\Delta E = \hbar^2 (2l+1)/2$. The two states in the same energy configuration but with the spin aligned or anti-aligned are called a doublet.

Example: Consider the N = 3 harmonic oscillator level. The level $1f_{7/2}$ is pushed far down (because of the high ***l***). Then its energy is so different that it makes a shell on its own. The nucleons which completely fill Shells upto N = 2 is 20, a third magic number. Then if we now consider the degeneracy of $1f_{7/2}$, D(***j***) = 2***j*** + 1 = 2(7/2) + 1 = 8, we obtain the 4th magic number 28.

Since the $1f_{7/2}$ level now forms a shell on its own and it does not belong to the N = 3 shell anymore, the residual degeneracy of N = 3 is just 12 instead of 20 as before. To this degeneracy, we might expect to have to add the lowest level of the N = 4 manifold. The highest ***l*** possible for N = 4 is obtained with n = 1 from the formula N = 2(n − 1) + ***l*** → ***l*** = 4 (this would be 1*g*). Then the lowest level is for ***j*** = ***l*** +1/2 = 4+1/2 = 9/2 with degeneracy D = 2(9/2)+1 = 10. This new combined shell comprises then 12+10 levels. In turns this gives us the magic number 50.

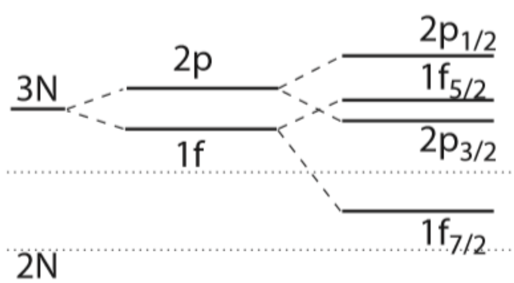

Fig. 1: The energy levels from the harmonic oscillator level (labeled by N) are first shifted by the angular momentum potential (2p, 1f). Each ***l*** level is then split by the spin-orbit interaction, which pushes the energy up or down, depending on the spin and angular momentum alignment

Using these same considerations, the splittings given by the spin-orbit coupling can account for all the magic numbers and even predict a new one at 184:

- $N = 4$, 1g $\rightarrow 1g_{7/2}$ and $1g_{9/2}$. Then we have $20 - 8 = 12 + D(9/2) = 10$. From 28 we add another 22 to arrive at the magic number 50.
- $N = 5$, 1h $\rightarrow 1h_{9/2}$ and $1h_{11/2}$. The shell thus combines the $N = 4$ levels not already included above, and the $D(1h_{11/2}) = 12$ levels obtained from the $N = 5$ $1h_{11/2}$. The degeneracy of $N = 4$ was 30, from which we subtract the 10 levels included in $N = 3$. Then we have $(30 - 10) + D(1h_{11/2}) = 20 + 12 = 32$. From 50 we add arrive at the magic number 82.
- $N = 6$, 1i $\rightarrow 1i_{11/2}$ and $1i_{13/2}$. The shell thus have $D(N = 5) - D(1h_{11/2}) + D(1i_{13/2}) = 42 - 12 + 14 = 44$ levels $(D(N) = (N + 1)(N + 2))$. The predicted magic number is then 126.
- $N = 7 \rightarrow 1j_{15/2}$ is added to the $N = 6$ shell, to give $D(N = 6) - D(1i_{13/2}) + D(1j_{15/2}) = 56 - 14 + 16 = 58$, predicting a yet not-observed 184 magic number.

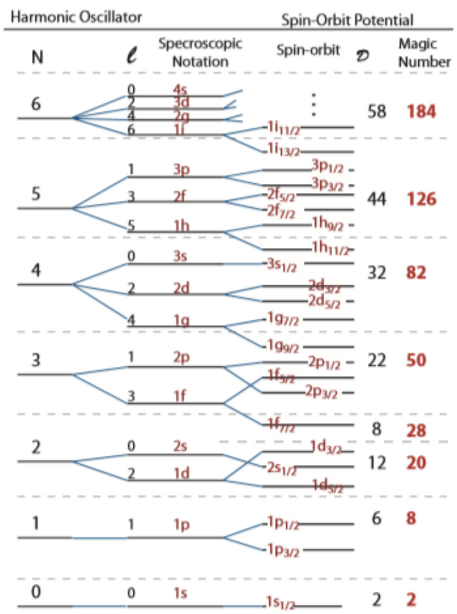

Fig. 2: Shell Model prediction of the magic numbers. Level splittings due to h.o. levels, ***l***-quantum number and spin-orbit coupling.

These predictions do not depend on the exact shape of the square well potential, but only on the spin-orbit coupling and its relative strength to the nuclear interaction V_0 as set in the harmonic oscillator potential. The shell model that we have just presented is quite a simplified model. However it can make many predictions about the nuclide properties. For example it predicts the nuclear spin and parity, the magnetic dipole moment and electric quadrupolar moment, and it can even be used to calculate the probability of transitions from one state to another as a result of radioactive decay or nuclear reactions.

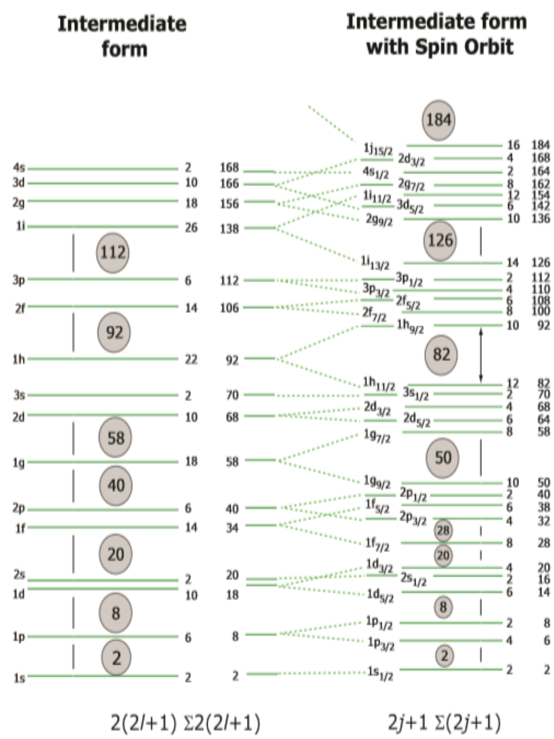

Fig. 3: Shell Model energy levels (from Krane Fig. 5.6). Left: Calculated energy levels based on potential. To the right of each level are its capacity and cumulative number of nucleons up to that level. The spin-orbit interaction splits the levels with $\boldsymbol{l} > 0$ into two new levels. Note that the shell effect is quite apparent, and magic numbers are reproduced exactly.

Nuclear spin and Parity

In the extreme shell model (or extreme independent particle model), the assumption is that only the last unpaired nucleon dictates the properties of the nucleus. A better approximation would be to consider all the nucleons above a filled shell as contributing to the properties of a nucleus. These nucleons are called the valence nucleons. Properties that can be predicted by the characteristics of the valence nucleons include the magnetic dipole moment, the electric quadrupole moment, the excited states and the spin-parity (as we will see). The shell model can be then used not only to predict excited states, but also to calculate the rate of transitions from one state to another due to radioactive decay or nuclear reactions. As the proton and neutron levels are filled the nucleons of each type pair off, yielding a zero angular momentum for the pair. This pairing of nucleons implies the existence of a pairing force that lowers the energy of the system when the nucleons are paired-off. Since the nucleons get paired-off, the total spin and parity of a nucleus is only given by the last unpaired nucleon(s) (which reside(s) in the highest energy level). Specifically we can have either one neutron or one proton or a pair neutron-proton. The parity for a single nucleon is $(-1)^l$, and the overall parity of a nucleus is the product of the single nucleon parity.

The shell model with pairing force predicts a nuclear spin I = 0 and parity Π =even (or $I^{\Pi} = 0^+$) for all even-even nuclides.

A. Odd-Even nuclei

Despite its crudeness, the shell model with the spin-orbit correction describes well the spin and parity of all odd-A nuclei. In particular, all odd-A nuclei will have half-integer spin (since the nucleons, being fermions, have half-integer spin).

Example: ${}^{15}_{8}O_7$ and ${}^{17}_{8}O_9$. (of course ${}^{16}O$ has spin zero and even parity because all the nucleons are paired). The first (${}^{15}_{8}O_7$) has an unpaired neutron in the $p_{1/2}$ shell, than $l = 1$, $s = 1/2$ and we would predict the isotope to have spin 1/2 and odd parity. The ground state of ${}^{17}_{8}O_9$ instead has the last unpaired neutron in the $d_{5/2}$ shell, with $l = 2$ and $s = 5/2$, thus implying a spin 5/2 with even parity. Both these predictions are confirmed by experiments.

Examples: These are even-odd nuclides (i.e. with A odd).

$\rightarrow$ ${}^{123}_{51}Sb_{72}$ has 1proton in $1g_{7/2}$: $\rightarrow$ $\frac{7}{2}^+$.

$\rightarrow$ ${}^{133}_{51}Cs$ has 1proton in $1g_{7/2}$: $\rightarrow$ $\frac{7}{2}^+$.

$\rightarrow$ ${}^{35}_{17}Cl$ has 1proton in $1d_{3/2}$: $\rightarrow$ $\frac{3}{2}^+$.

$\rightarrow$ ${}^{29}_{14}Si$ has 1 neutron in $2s_{1/2}$: $\rightarrow$ $\frac{1}{2}^+$.

$\rightarrow$ ${}^{28}_{14}Si$ has paired nucleons: $\rightarrow$ 0^+.

Example: There are some nuclides that seem to be exceptions:

$\rightarrow$ ${}^{121}_{51}Sb_{70}$ has last proton in $2d_{5/2}$ instead of $1g_{7/2}$: $\rightarrow$ $\frac{5}{2}^+$ (details in the potential could account for the inversion of the two level order)

$\rightarrow$ ${}^{147}_{62}Sn_{85}$ has last proton in $2f_{7/2}$ instead of $1h_{9/2}$: $\rightarrow$ $\frac{7}{2}^-$.

$\rightarrow$ ${}^{79}_{35}Br_{44}$ has last neutron in $2p_{3/2}$ instead of $1f_{5/2}$: $\rightarrow$ $\frac{3}{2}^-$.

$\rightarrow$ ${}^{207}_{82}Pb_{125}$. Here we invert $1i_{13/2}$ with $3p_{1/2}$. This seems to be wrong because the 1i level must be quite more energetic than the 3p one. However, when we move a neutron from the 3p to the 1i all the neutrons in the 1i level are now paired, thus lowering the energy of this new configuration.

$\rightarrow$ ${}^{61}_{28}Ni_{33}$ $1f_{5/2} \longleftrightarrow 2p_{3/2} \rightarrow (\frac{3}{2}^-)$

$\rightarrow$ ${}^{197}_{79}Au_{118}$ $1f_{5/2} \longleftrightarrow 3p_{3/2} \rightarrow (\frac{3}{2}^+)$

B. Odd-Odd nuclei

Only five stable nuclides contain both an odd number of protons and an odd number of neutrons: the first four odd-odd nuclides ${}^{2}_{1}H$,${}^{6}_{3}Li$, ${}^{10}_{5}B$, and ${}^{14}_{7}N$. These nuclides have two unpaired nucleons (or odd-odd nuclides), thus their spin is more complicated to calculate. The total angular momentum can then take values between $|j_1 - j_2|$ and $j_1 + j_2$.
Two processes are at play:
1) the nuclei tends to have the smallest angular momentum, and
2) the nucleon spins tend to align (this was the same effect that we saw for example in the deuteron In any case, the resultant nuclear spin is going to be an integer number.